A Mathematical Introduction to the Game of Minesweeper

Philip Crow
Mathematics Dept.
Southern Nazarene University
Bethany, OK 73008
pcrow@snu.edu

Introduction

We construct a systematic notation for the one-person game of Minesweeper that has been made popular by its availability on various recent computers. We use the notation in proving several theorems that can aid players during the game, and we provide patterns for generating additional results.

The game board consists of a rectangular array of squares. At the start of the game, the contents of each square is concealed. When a specified action is performed (usually a mouse click of a square), the contents of the selected square is revealed. The square may contain a mine, in which case the game ends (and the player loses). Otherwise, a number 0 through 8 appears in the square, denoting the number of mines in squares adjacent to the revealed square. If the player believes that a square contains a mine, the player may mark the square as mined. The total number of mines is known in advance of play. The goal of the game is to mark all of the mines and clear (by clicking them) all of the other squares.

Notation

Let the lower-left square of the board be denoted by $(0, 0)$, and number the rows above it consecutively from 1 onward, and the columns to the right consecutively from 1 onward. Denote the coordinates of square A in this system by (X_A, Y_A) (see **Figure 1**).

For some purposes, we will use not these absolute coordinates but coordinates relative to a given square. Given a square A, the *A-ocentric coordinates* of a square B are $(X_B - X_A, Y_B - Y_A)$. These new coordinates represent taxi directions from A to B (see **Figure 2**).

The UMAP Journal 18 (1) (1997) 35–42.

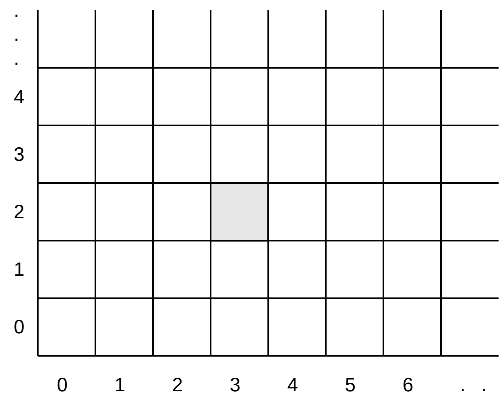

Figure 1. Standard board numbering. The shaded square is represented as $(3, 2)$.

Let

$N(A)$ ("number") be the number of mines in squares adjacent to A,
$K(A)$ ("knowledge") be the set of squares adjacent to A,
$C(A)$ ("clear") be the members of $K(A)$ that are known to contain no mines, and
$M(A)$ ("mined") be the members of $K(A)$ that are known to contain a mine.

For a square A not on the edges of the grid, $K(A)$ consists of eight squares, which can be listed in A-ocentric coordinates as

$$K(A) = \{(-1,-1),(-1,0),(-1,1),(0,1),(1,1),(1,0),(1,-1),(0,-1)\}.$$

Note that membership in $C(A)$ and $M(A)$ is based on the player's knowledge. A square is only in $M(A)$ if the player is certain, by logical deduction, that the square contains a mine. At any given time, there may be squares that are in $K(A)$ but are in neither $C(A)$ nor $M(A)$.

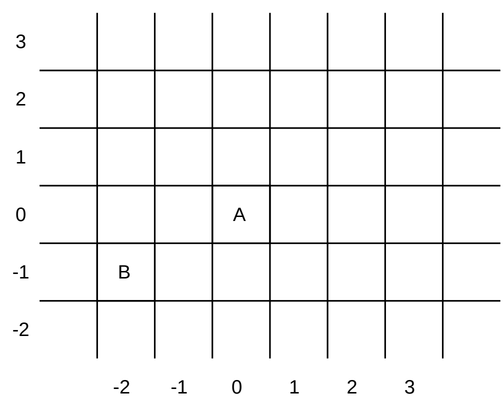

Figure 2. A-ocentric board numbering. The coordinates of B are now given as $B^A = (-2, -1)$.

Results

The Basic Facts: Exhaustion

- *If only $N(A)$ squares adjacent to A remain concealed, these squares have mines.*
- *If $N(A)$ mines have already been marked adjacent to A, the remaining squares adjacent to A are clear.*

These facts allow us to move squares into our knowledge sets $C(A)$ and $M(A)$; they are the fundamental facts of the game.

For two squares A and B, define

$$K(AB) = K(A) \cap K(B).$$

If squares A and B are horizontally adjacent, with A on the left, then we can express the set of squares adjacent to B in A-ocentric coordinates as

$$K(B^A) = \{(-2,-1),(-2,0),(-2,1),(-1,1),(0,1),(0,0),(0,-1),(-1,-1)\}.$$

For this situation, we have

$$K(AB) = \{(-1,-1),(-1,1),(0,1),(0,-1)\}.$$

The 1–4 Theorem. *Let A and B be horizontally adjacent with A on the left. If $N(A) = 4$ and $N(B) = 1$, then, in B-ocentric coordinates,*

- *each member of $K(A) \setminus (A \cup K(AB)) = \{(-2,-1),(-2,0),(-2,1)\}$ contains a mine,*
- *exactly one member of $K(AB) = \{(-1,-1),(-1,1),(0,1),(0,-1)\}$ contains a mine, and*
- *there are no mines in $K(B) \setminus (A \cup K(AB)) = \{(1,1),(1,0),(1,-1)\}$.*

Figure 3 gives a pictorial summary of this result, with the conventions:

- A number indicates that a square has already been revealed; the number is the number of adjacent mines.
- A question mark indicates a square of undetermined contents.
- An asterisk indicates a square known to contain a mine.
- A "V" indicates a square that cannot contain a mine.

	-2	-1	0	1	2
2					
1	*	?	?	V	
0	*	4	1	V	
-1	*	?	?	V	
-2					

Figure 3. The facts known when row adjacent squares have 1 and 4 adjacent mines, respectively.

Proof: Since $N(B) = 1$ exactly one member of $K(B)$ contains a mine. So, any subset of $K(B)$ contains at most one mine. In particular, this implies that $K(AB)$ contains at most one mine. Since no mine is located at B and $K(AB)$ contains at most one mine, $K(A) \setminus (B \cup K(AB))$ contains at least three mines (since $N(A) - 1 = 3$). But the size of that set is three, thus every member must contain a mine.

We have now placed three members into $M(A)$. Exactly one member remains; and since B is clear, the mine must be in $K(AB)$. By exhaustion, there cannot also be a mine in $K(B) \setminus (A \cup K(AB))$. So that set is free from mines.$\square$

Unfortunately, the hypothesis of the 1–4 Theorem is not often satisfied in practice. The following situation (see **Figure 4**) arises more often.

	-2	-1	0	1	2
2					
1	V	?	V	?	V
0	V	1	2	1	V
-1	V	?	V	?	V
-2					

Figure 4. The facts known when row adjacent squares have 1, 2, and 1 adjacent mines, respectively.

The 1-2-1 Theorem. *Let A, B, and C be consecutive squares in a row. If $N(A) = N(C) = 1$ and $N(B) = 2$, then, in B-ocentric coordinates,*

- $(0, 1)$ *and* $(0, -1)$ *are clear,*
- $K(B) \setminus (A \cup K(AB)) = \{(-2, -1), (-2, 0), (-2, 1)\}$ *contains no mines, and*
- $K(C) \setminus (B \cup K(AB)) = \{(2, -1), (2, 0), (2, 1)\}$ *contains no mines.*

Proof: Assume that (0,1) contains a mine. Then since $N(A) = 1$, exhaustion implies that there are no mines in $K(A) \setminus (0, 1)$. Exhaustion further implies that $K(C) \setminus (0,1)$ contains no mines. But then there would be no square left in $K(B)$ in which to place the second mine required by $N(B)$. This contradiction proves that (0,1) cannot contain a mine. By symmetry, $(0,-1)$ cannot contain a mine.

Since $N(C) = 1$, squares $(1,1)$ and $(1,-1)$ cannot both contain mines. From the first result of this theorem and the fact that squares A and C are themselves clear, squares $\{(-1,-1),(-1,1)\}$ must contain exactly one mine. Hence $\{(1,1),(1,-1)\}$ must also contain exactly one mine. The claimed results follow by exhaustion. □

An *open n-row* is a row of n consecutive squares for which the n consecutive squares *below* them are known to contain no mines.

Lemma. *If A, B, C are consecutive squares in an open horizontal 3-row, then $N(B) \leq 3$.*

The Open 3 Theorem. *Let A, B, C be consecutive squares in an open horizontal 3-row. If $N(B) = 3$, then, in B-ocentric coordinates, squares $(-1,1)$, $(0,1)$, and $(1,1)$ all contain mines.*

This result is shown pictorially in **Figure 5**. Here an "N" indicates that the square has been revealed (so it is not a mine). The number of mines adjacent to such a square does not figure in the result.

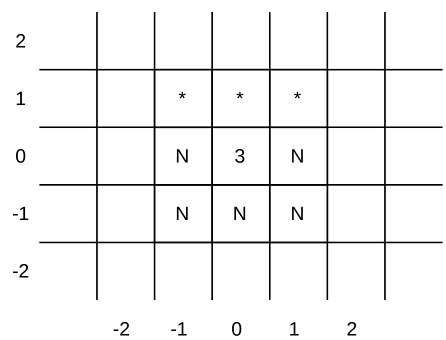

Figure 5. The facts known when a square in an open row has 3 adjacent mines.

The Lemma and the Open 3 Theorem follow directly from the definition of open rows and exhaustion.

The Open 1–2–1 Theorem. *Let A, B, C be consecutive squares in an open horizontal 3-row. If $N(A) = N(C) = 1$ and $N(B) = 2$, then, in B-ocentric coordinates, squares $(-1,1)$ and $(1,1)$ contain mines.*

This result follows directly from the definition of open rows and the 1–2–1 Theorem. It is shown pictorially in **Figure 6**.

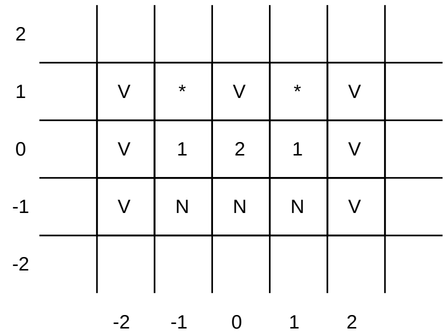

Figure 6. The facts known when adjacent squares in an open row have 1, 2, and 1 adjacent mines, respectively.

The last two results are the beginning of a list of extremely useful facts for actually playing the game. **Figure 7** shows several of these additional results. The last six involve the edge of the board. Since the edge of the board merely increases our knowledge of where mines might be, there are equivalent theorems for areas removed from the edge if the same extra knowledge is in hand. In fact, two of the theorems shown in the figure are equivalent (which two?).

Such a list of results can be greatly expanded. Let $R(A) = N(A) - |M(A)|$ be the number of mines adjacent to A that have not yet been identified. All of the above theorems hold when $N(\cdot)$ is replaced by $R(\cdot)$ if the members of $M(A)$ are removed from consideration. To see that this is true, either re-prove each theorem, or realize that it is only outstanding mines on which the results hang. For an example, consider the "relative" 1–2–1 theorem shown in **Figure 8**.

Our results have been couched in terms of rows, and the definition of open n-row refers to the row below the given row; but the game is symmetric as far as rows and columns are concerned, and with regard to above / below and left / right.

Some additional open questions concern probability results, such as what is the best beginning tactic.

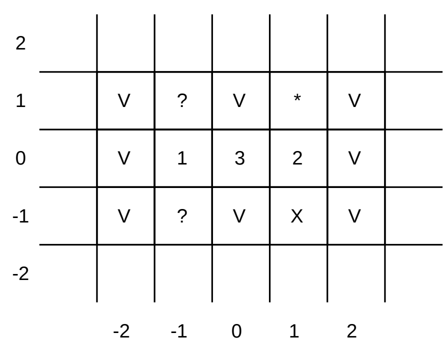

Figure 8. One example of three squares in a row that have relative mine counts of 1, 2, and 1. An "X" indicates a mine known in advance, and asterisks are mine locations given by the relative 1-2-1 Theorem.

V	?	?	*
V	1	3	*
V	N	N	N

V	?	V	*	V
V	1	3	2	V
V	?	V	*	V

V	?	?	*
V	1	2	N
V	N	N	N

V	V	?	?	V	V
V	1	2	2	1	V
V	V	?	?	V	V

V	?	?	N
V	1	1	N
V	?	?	N

N	N	*
N	1	N
N	N	N

V	V	V
?	1	?
?	1	?

V	V	V
?	2	?
?	2	?

V	V	V
?	3	?
?	3	?

V	V	V
*	4	*
*	4	*

N	1	?	V
N	1	1	V
N	1	?	V
N	1	?	?

N	1	?	V	V
N	1	1	1	V
N	1	?	1	V
N	1	?	V	V

Key:
N - Any number is showing.
1,2,3, 4 - The given number must be showing.
? - You know nothing about this square.
* - This square has a mine.
V - This square does not have a mine.
—— - The edge of the board.

Figure 7. A summary of other Minesweeper theorems. The key is slightly expanded to include a thick black line that indicates the edge of the board.

Acknowledgments

The author wishes to express thanks to Robert Stutts and Scott Smith of Columbia College, SC, for suggesting this problem, and to Lee Turner of his own department for encouragement in pursuing it.

About the Author

Philip Crow received a B.A. from Mid-America Nazarene College (mathematics) and a Ph.D. from the University of Virginia (applied mathematics). He is an assistant professor of mathematics at Southern Nazarene University, where he is in his third year.